# VARIATIONAL ITERATION METHOD FOR SANDWICH PANEL STABILITY

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## ABSTRACT

The web core sandwich structure subjected to compressive loads perpendicular to the webs has the susceptibility of buckling within a unit cell. The buckling behavior of the unit cell under com- pression loading can be modeled as the elastic buckling of columns resting on a two-parameter Pasternak foundation with rotational restraints at two ends. In this paper, the effects of the Pasternak foundation and rotational end restraints existing simultaneously on the critical buckling load are investigated. An analytical approximation technique, variational iteration method (VIM) is applied. Based on solving the characteristic equation, exact solutions are also presented to validate the VIM solutions. The results indicate the great significance of elastic foundations in increasing the stability. The effects of boundary conditions on critical buckling load are trivial only when stiff foundations are used. The determination of foundation parameters are proposed and evaluated. The importance of web-pitch to face-sheet thickness ratio is found. Longer web-pitches are desired to increase critical loads, which also help the structure weight optimization since fewer webs are needed for a given total width. The novelties of current work include the application of variational iteration algorithm to the problem, the investigations and comparisons of combinational effects of elastic foundations and rotational restraints, the evaluations of foundation parameters based on practical materials and useful suggestions on the structure design. The exact solutions can serve as benchmarks for other numerical methods.

## 1 INTRODUCTION

The. The buckling of columns on elastic foundations is drawing wide interests from many researchers in various fields since it represents numerous practical applications [1–4]. The exploration of column buckling has a long history and has been systematically explained in well-known literatures [5–7]. After the Winkler elastic foundation model first proposed [8], more sophisticated and practical foundation models were presented [9–12], one of which, the Pasternak model, was demonstrated to be applicable to many problems.

Focusing on the column resting on elastic foundations, Sundararajan [13] studied the stability problem of columns on elastic foundations subjected to conservative and non-conservative forces. The Winkler's model was used and the influences of the foundation were investigated. A finite element method for the vibration of beam-column on two-parameter elastic foundation was presented by Yokoyama [14]. The finite element method was shown to be effective, and comprehensive parameter studies were then performed. Morfidis and Avramidis [15] proposed a generalized finite element for the beam-column on elastic foundations. Effects of shear deformations, semi-rigid connections, rigid offsets and axial forces could be included in the elements. A two-parameter elastic foundation model was used in their research. Post-buckling analysis of an elastic column on the Winkler foundation was performed with the employment of an approximate analytic technique [16]. The responses of perfect and geometrically imperfect columns were discussed.

For beams and plates on elastic foundations, Feng and Robert [17] suggested a finite element method to analyze beams on two-parameter foundations. Two types of beam elements are formulated and com- pared. It is shown that elements based on the exact displacement function predict the results more accurately and are computationally cheaper. Levy [18] proposed a weight minimization method for beams and plates on elastic foundations for given buckling loads and optimality criteria is derived using the variational method. The buckling of simply supported laminates on Pasternak foundations subjected to uniaxial and biaxial in-plane loads is investigated by Xiang *et al* [19]. The first-order shear deformation plate theory was employed in their research. Calculus of variations was applied to minimize the total potential energy functional and the characteristic eigenvalue equation is derived based on the Navier method. Numerical results are obtained, based on which, comprehensive parameter studies are conducted. Lam *et al* [20] presented canonical exact solutions for elastic bending, buckling and vibration of isotropic plates on two-parameter foundations. Green's functions were used in the paper and the plates were limited to the Levy type. Web core sandwich panels under in-plane compression were analyzed and optimized for the minimum weight considering instability failure criteria in [21]. The web boundaries of each unit cell is assumed to be simple support to provide conservative results and the core is modeled as a one-parameter elastic foundation, which is modeled as linear elastic spring. The effects of foundations are clearly demonstrated. Similar research was carried out by Yu [22] for Levy plates on a one-parameter foundation. Exact solutions are obtained for both uniaxial and biaxial loads. Buckling of steel beam column on Pasternak foundations with simply supported - simply supported and clamped- clamped boundary conditions are investigated in [23]. The high order mode coupling is found and is symbolically determined for the former boundary condition. In terms of the determination of foundation parameters, Sironic [24] reevaluated the foundation constants using the Airy stress function with the plane strain assumption and the principle of minimum total potential energy. Modified foundation parameters are suggested for deep and shallow elastic foundations. Recently, Briscoe [25] examined the shear buckling of isotropic plates on Pasternak foundations. A new model for the foundation parameters is proposed with the application minimum total potential energy principle.

Variational iteration method (VIM) is powerful in solving problems related to differential equations. The buckling of non-uniform column with rotational end restraints are investigated with the application of VIM [26], which is demonstrated to be an efficient tool to solve differential equation and boundary value problems [27, 28]. The same method is used in the research on the buckling of the Euler column with continuous elastic restraints [29]. The continuous elastic restraints are modeled as elastic linear springs and several combinations of boundary conditions are investigated.

It can be seen from the literature review above that combinational effects of rotational restraints and elastic foundations, which is common and realistic in engineering structures, has not been investigated. In this paper, the buckling analysis of columns on two-parameter Pasternak foundations with rotational end restraints are performed. Due to the complexity of the problem, no explicit analytical solutions are available. The characteristic eigenvalue equation for the column buckling is then derived and evaluated numerically. The variational iteration method is used to validate the numerical results. The numerical results can serve as benchmarks for further approximate and numerical solutions. The effects of rotational spring stiffness, elastic foundation parameters and the combined effects of both are demonstrated. Parameter studies on foundation parameters are presented. Furthermore, the solutions are applied to the buckling of web core sandwich panel subjected to compressive loads normal to webs.

# 2 BUCKLING MODEL

Consider an Euler beam column of length $L$ and thickness $t$ resting on a two-parameter foundation with rotational springs of stiffness constants $k$, acting on two ends, Figure 1. The elastic modulus of the column is , the second moment of area is $I$ and the flexural rigidity is $D = EI$. The two foundation parameters are $K_w$, the Winkler foundation parameter, which describes the foundations as a series of linear elastic springs normal to the beam, and

$K_p$ which describes the interactions between springs. The beam column is subjected to compressive force, $P$. According to [23, 30], the deflection of the beam $\overline{w}(\bar{x})$ is governed by

$$\mathrm{D}\frac{d^4}{d\bar{x}}\overline{w}(\bar{x}) + P\frac{d^2}{d\bar{x}^2}\overline{w}(\bar{x}) + K_w\overline{w}(\bar{x}) - K_p\frac{d^2}{d\bar{x}^2}\overline{w}(\bar{x}) = 0 \tag{1}$$

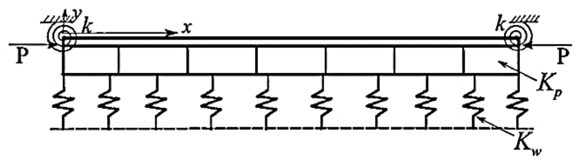


Figure 1. beam model on the elastic foundation.

For computation ease and convenience, the equation above is rewritten in the non-dimensional form as

$$\frac{d^4}{dx^4}w(x) + \pi^2(p - \kappa_p)\frac{d^2}{dx^2}w(x) + \pi^4\kappa_w w(x) = 0 \tag{2}$$

where $w$ is the function with respect to $x(x = \bar{x}/L)$. Other non-dimensional parameters shown in the equation above are

$$p = \frac{PL^2}{\pi^2 D} \qquad \kappa_p = \frac{K_p L^2}{\pi^2 D} \qquad \kappa_w = \frac{K_w L^4}{\pi^4 D} \tag{3}$$

For the non-dimensional differential equation, the boundary conditions at $x = 0$ are

$$\frac{d^2}{dx^2}w(0) = \kappa\frac{d}{dx}w(0), \qquad w(0) = 0 \tag{4}$$

and the boundary conditions at $x = 1$ are

$$\frac{d^2}{dx^2}w(1) = -\kappa\frac{d}{dx}w(1), \qquad w(1) = 0 \tag{5}$$

The non-dimensional parameter κ in Equation 4 and 5 is

$$\kappa = \frac{kL}{D} \tag{6}$$

According to [7], the general solution to equation 2 is

$$w(x) = C_1\csc(\beta_1 x) + C_2\sin(\beta_1 x) + C_3\csc(\beta_2 x) + C_4\sin(\beta_2 x) \tag{7}$$

where

$$\beta_1 = \sqrt{\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - \xi}} \qquad \beta_2 = \sqrt{\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - \xi}} \tag{8}$$

$$\alpha = \pi^2(p - \kappa_p) \qquad \xi = \pi^4\kappa_w \tag{9}$$

Boundary conditions are used to determin Constants $C_1$ to $C_4$. Equation 8 holds when $\frac{\alpha^2}{4} - \xi \geq 0$, which gives $p \geq \kappa_p + 2\sqrt{\kappa_w}$. Substituting in Equation 4 and 5 with Equation 7 leads to the characteristic equation.

# 3 VARIATIONAL ITERATION METHOD

VIM is an analytical approximation technique. It is widely used in solving nonlinear differential equations with the advantages of effectiveness, accuracy and converging to exact solutions rapidly [28]. Considering a homogeneous nonlinear differential system as follows:

$$\mathrm{L}[\mathrm{w}(\mathrm{t})] + \mathrm{N}[\mathrm{w}(\mathrm{t})] = 0 \tag{10}$$

where $L$ is a linear operator and $N$ is a nonlinear operator.

To solve the nonlinear differential equation above using VIM, a correction function should be con- structed. According to He *et al* [28], three iteration formulas are commonly used, including

$$w_{n+1}(x) = w_n(x) + \int_0^x \lambda(\zeta)(L[w_n(\zeta)] + N[\tilde{w}_n(\zeta)])d\zeta \tag{11}$$

$$w_{n+1}(x) = w_0(x) + \int_0^x \lambda(\zeta)N[w_n(\zeta)]d\zeta \tag{12}$$

$$w_{n+2}(x) = w_{n+1}(x) + \int_0^x \lambda(\zeta)(N[w_{n+1}(\zeta)] - N[w_n(\zeta)])d\zeta \tag{13}$$

where $\lambda$ is a general Lagrange multiplier that can be identified optimally via variational theory, $w_0$ is the initial guess and $w_n$ is the n-th approximate solution and $\tilde{w}_n$ denotes a restricted variation [27, 28]. Equation 11, 12 and 13 are variational iteration algorithm I, II and III, respectively. The initial guess $w_0$ in algorithm II is required to satisfy the boundary conditions, which is complicated in the present problem due to the existence of the restraints at the ends. Thus, the simpler algorithm I is chosen. For a four order differential equation, a simple Lagrange multiplier is suggested in [26] as

$$\lambda(\zeta) = \frac{(\zeta - x)^3}{6} \tag{14}$$

With the Lagrange multiplier, the correction function for present problem is represented as

$$\mathrm{w}_{n+1}(x) = w_n(x) + \int_0^x \frac{(\zeta - x)^3}{6}\left(\frac{d^4}{d\zeta^4} w_n(\zeta) + \pi^2(p - \kappa_p)\frac{d^2}{d\zeta^2} w_n(\zeta) + \pi^4\kappa_w w_n(\zeta)\right)d\zeta \tag{15}$$

The initial solution $w_0$ of the deflection function of the beam can be freely selected and unknown parameters can be contained in it. The initial solution is chosen to be a polynomial, which is

$$w_o(x) = Ax^3 + Bx^2 + Cx + D \tag{16}$$

With the initial solution and the correction function, iterations can be conducted. MATLAB is used to facilitate computations. After the nth iterations, an approximate solution is obtained, which will be substituted into the boundary conditions, Equation 4 and 5. Correspondingly, four homogeneous equations are obtained from the four boundary conditions and the characteristic equation is derived by making the determinant of coefficient matrix of the four homogeneous equations zero. The accuracy of VIM is related to iteration times.

# 4 NUMERICAL EVALUATIONS AND DISCUSSION

Numerical evaluations of the characteristic equation from Section 2 and the characteristic equation of the approximate solution from VIM are performed. The VIM procedures are implemented in MATLAB and the critical load is easily found by MATLAB. As mentioned above, no literatures have been published regarding this problem. A special cases, i.e. column with rotational end restraints without elastic foundation is evaluated and compared with the results in the literature. The numerical evaluation of the case is achieved by setting the foundation parameters to zero. In this paper, the rotational restraints at two ends are made equal, which is practical for most web core sandwich structures. Obviously, non-equal restraints situations can also be calculated using the two method presented in this paper.

It can be seen from Table 1 that the present exact solutions are exactly the same as those in [7]. The approximate analytical solutions after 20 iterations using VIM are close to the exact solutions with high degree of accuracy. Then, buckling solutions of column on Winkler's and Pasternak's foundations are found using both methods and the results are shown in Table 2 and 3. The range of normalized stiffness constants of rotational restraints is from 0.1 to infinity ($10^9$). Different ratios of two foundation parameters are chosen in Table 3. The computation of 25 iterations is conducted in the VIM. Identical or extremely close results are yielded from both methods.

| $\boldsymbol{\kappa}$ | VIM(n=10) | Wang [**7**]† |
|---|---|---|
| 0 | 1 | 1 |
| 0.5 | 1.1927 | 1.1928 |
| 1 | 1.3671 | 1.3670 |
| 2 | 1.6681 | 1.6681 |
| 4 | 2.1234 | 2.1234 |
| 10 | 2.8540 | 2.8540 |
| ∞ | 3.9999 | 4 |

†Due to different method of normalization, the values in the table are obtained by dividing the results in [7] by $\boldsymbol{\pi^2}$

Table 1. Exact and approximate solutions (Winkler's foundations)

| | | $\kappa_w$=1, $\kappa_p$=0 | $\kappa_w$=10, $\kappa_p$=0 | $\kappa_w$=100, $\kappa_p$=0 | $\kappa_w$=150, $\kappa_p$=0 | $\kappa_w$=300, $\kappa_p$=0 |
|---|---|---|---|---|---|---|
| $\kappa = 0.1$ | Exact | 2.0401 | 6.5402 | 20.1513 | 25.4152 | 34.7902 |
| | VIM | 2.0401 | 6.5402 | 20.1513 | 25.4152 | 34.7956 |
| $\kappa = 1$ | Exact | 2.3644 | 6.8734 | 20.4815 | 25.7472 | 35.1213 |
| | VIM | 2.3644 | 6.8734 | 20.4815 | 25.7472 | 35.1231 |
| $\kappa = 10$ | Exact | 3.7658 | 8.5731 | 22.0763 | 27.2434 | 36.7527 |
| | VIM | 3.7658 | 8.5731 | 22.0763 | 27.2434 | 36.7130 |
| $\kappa = 10^2$ | Exact | 4.6157 | 9.9614 | 23.2506 | 27.8283 | 38.0160 |
| | VIM | 4.6157 | 9.9614 | 23.2506 | 27.8283 | 37.9560 |
| $\kappa = 10^4$ | Exact | 4.7419 | 10.1973 | 23.4401 | 27.9069 | 38.2264 |
| | VIM | 4.7419 | 10.1973 | 23.4401 | 27.9069 | 38.1754 |
| $\kappa = \infty$ | Exact | 4.7432 | 10.1998 | 23.4420 | 27.9077 | 38.2286 |
| | VIM | 4.7432 | 10.1998 | 23.4420 | 27.9078 | 38.1778 |

Table 2. Exact and approximate solutions (Winkler's foundations)

### *4.1 Effects of Rotational End Restraints and Foundation Parameters*

A parameter study on the normalized rotational spring constant $\kappa$ is presented first. For the sack of brevity, the elastic foundation is absent and the normalized critical load is found for different normalized rotational spring constants, as illustrated in Figure 4. To show the results explicitly, the $\log\kappa$ scale is generated for the $x$ axis. Apparently, the normalized critical load increases slowly when $\kappa$ exceeds 100 ($\lg\kappa = 2$). When the normalized rotational spring constant is 10000, the normalized critical load is almost 4 (3.9986 form the exact solution and 4.006 from VIM), which is the normalized critical load for the clamped-clamped boundary condition. The critical load merely increases when $k$ exceeds 10000. In the later part of this paper, $\kappa = 10^9$ is regarded as the clamped-clamped boundary condition to facilitate computations and explanations. This will hold for both the columns with or without elastic foundations with the assumption that the rigidity of rotational springs is not influenced by the existence of core.

The combinational effects of rotational end restraints and foundation parameters are demonstrated then. Normalized rotational spring constants from 1 to 10000 and various foundation parameter ratios, i.e. $\kappa_w/\kappa_p = 5$, 15 and 25 with the range 0 to 300 of $\kappa_w$, are covered to achieve generalities. The numerical results are shown in Figure 5, 6 and 6. Obviously, The incorporation of foundations increases the critical buckling loads dramatically. The four lines representing different normalized rotational spring constants are close to each other, which is true for all the three different foundation parameter ratios. The lines are approaching linear and parallel to each other with the increase of foundations parameters. It implies that the effects of foundation parameters are dominant, while rotational restraints have more significant effects when the foundations are weaker. To further demonstrate the effect of rotational springs for various foundation parameters, the increasing percent of critical load compared with column on elastic foundation with pin-pin boundary conditions are shown in Figure 8.

| | | $\kappa_w = 1, \kappa_p = 0.01$ | $\kappa_w = 10, \kappa_p = 2$ | $\kappa_w = 100, \kappa_p = 10$ | $\kappa_w = 150, \kappa_p = 50$ | $\kappa_w = 300, \kappa_p = 50$ |
|---|---|---|---|---|---|---|
| $\kappa = 0.1$ | Exact | 2.0501 | 8.5402 | 30.1513 | 75.4152 | 84.7902 |
| | VIM | 2.0501 | 8.5402 | 30.1513 | 75.4152 | 84.7956 |
| $\kappa = 1$ | Exact | 2.3744 | 8.8734 | 30.4815 | 75.7472 | 85.1213 |
| | VIM | 2.3744 | 8.8734 | 30.4815 | 75.7472 | 85.1231 |
| $\kappa = 10$ | Exact | 3.7758 | 10.5731 | 32.0763 | 77.2434 | 86.7529 |
| | VIM | 3.7758 | 10.5731 | 32.0763 | 77.2434 | 86.7130 |
| $\kappa = 10^2$ | Exact | 4.6257 | 11.9614 | 33.2506 | 77.8283 | 88.0160 |
| | VIM | 4.6257 | 11.9614 | 33.2506 | 77.8283 | 87.9560 |
| $\kappa = 10^4$ | Exact | 4.7519 | 12.1973 | 33.4401 | 77.9069 | 88.2264 |
| | VIM | 4.7519 | 12.1973 | 33.4401 | 77.9069 | 88.1754 |
| $\kappa = \infty$ | Exact | 4.7532 | 12.1998 | 33.4420 | 77.9077 | 88.2286 |
| | VIM | 4.7532 | 12.1998 | 33.4420 | 77.9078 | 88.1777 |

Table 3. Exact and approximate solutions (Pasternak's foundations)

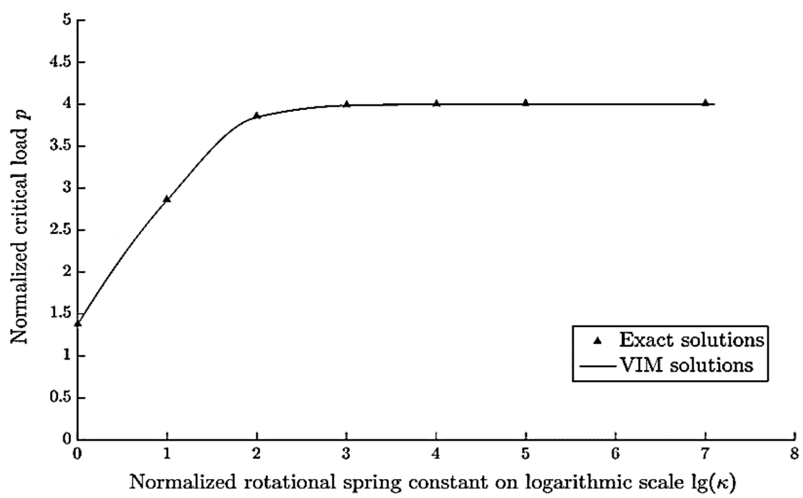


Figure 2. Column without elastic foundations.

The increase of critical load is significant, over 70%, for small foundation parameters while that for large foundation parameters is neglectable since the maximum increase is around 5%.

# 5 APPLICATION TO WEB CORE SANDWICH STRUCTURE

The sandwich structure is extensively used in many engineering industries, such as aerospace, ocean and building industry, due to the advantageous properties of high stiffness, light weight and design effectiveness [31]. Web core sandwich structures consist of two face-sheets connected and supported by interior webs, and core bonded to the face-sheets and webs. Web core sandwich panel have been applied to large ship structures and residential building roof to satisfy special requirements [2, 25, 32]. The improvement of shear property and fatigue life of web core sandwich structures can be achieved with the use of core materials [33, 34].

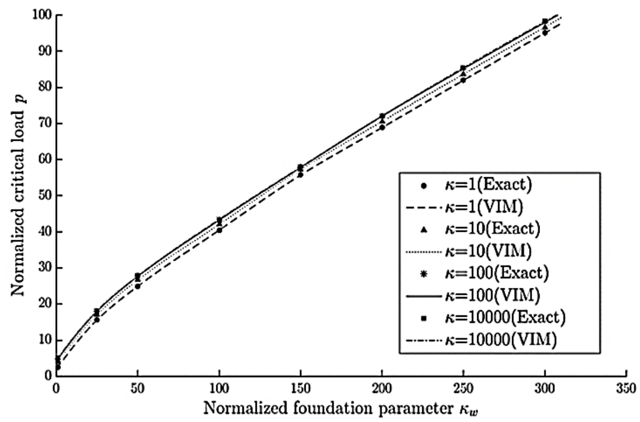


Figure 5. Buckling of column on elastic foundations $(\kappa_w/\kappa_p = 5)$

It is believed modeling the webs as simply-supported or clamped boundary conditions are simplistic and introduces significant errors. A practical method is to discrete webs and face-sheets and model webs as rotational restraints. The effect of rotational restraints on the buckling behavior of plate and beam has drawn the attention of many researchers. Lundquist and Stowell [35] obtained the exact and approximate solutions for the buckling of isotropic plates subjected to uniaxial compression and rotationally constrained along unloaded edges. Valuable data on critical buckling stresses are provided. Bleich [36] investigated the buckling of box shape under compression. The formulas of rotational constraint stiffness from the two sides of the box shape are presented. Explicit solutions for the buckling of orthotropic plate with rotational restraints using Ritz method are presented in [37–39]. The solution is applied to I section, C section, and box section *etc*. The formula to determine constraint stiffness constant in [36] is extended to orthotropic plates in their work. Significant effects of the rotational restraints on local bucking are found. Similar work on the buckling of rotational restrained fiber reinforced plastic composite plates are conducted by Kollar [40, 41]. More details concerning local buckling with rotational constraints can also be found in [42]. Furthermore, the rotational constraints are considered for the laser-welded web core sandwich plate [43]. The method to determine the rotational spring stiffness is proposed for laser welding. Linear spring and rotational spring are combined to model the general boundary conditions in [44]. Euler beam buckling with general boundary conditions are examined using Galerkin method.

The web core panel subjected to compression and bending loads is susceptible to local buckling [2], Figure 9. When a panel is subjected to uniformly distributed compressive loads perpendicular to webs with unloaded edges free, the constrained buckling of face-sheet, which is referred to as the buckling of face-sheets between webs, may occur [2]. Due to the periodicity, the whole panel is represented by a unit cell, Figure 10. The webs in the unit cell provide rotational restraints to the face-sheet and the core acts as the elastic foundation. Hence, the constrained buckling of web core sandwich panel resembles the buckling of column on elastic foundations with rotational ends restraints. The column is of unit width and accounts for Poisson's ratio effects [45], which means the flexural rigidity should be modified as $D_p = EI/(1-\upsilon^2)$.

### *5.1 Evaluation of Foundation Parameters of Typical Core Material*

The foundation parameters have significant effects on the critical buckling load. Therefore, the foundation parameters are further evaluated to give a insight of them and the evaluations are based on practical web core sandwich geometries. Research on the determination of the two foundation parameters are available in [24, 46]. In many web core sandwich panels, the thickness of core is relatively small compared to the web-pitch, which means the core is shallow. For shallow foundations, the determination of parameters is provided in [47] and different equations are proposed in [2]. The latter, as Equation 17, is used here because they are validated by finite element analysis. Normalized foundation parameters are expanded and expressions with respect to the structure geometries and material properties are obtained, Equation 18,

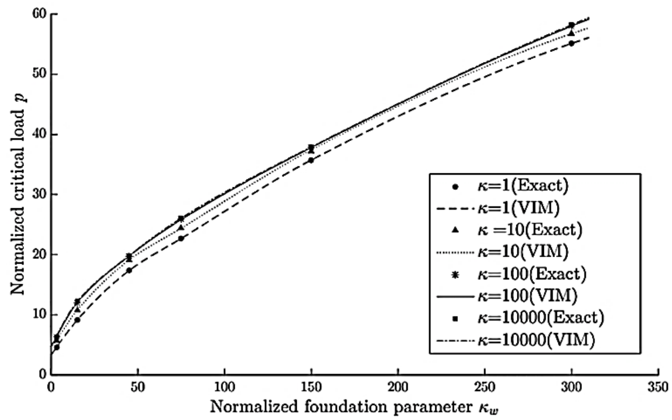


Figure 3. Buckling of column on elastic foundations ($\kappa_w/\kappa_p = 15$).

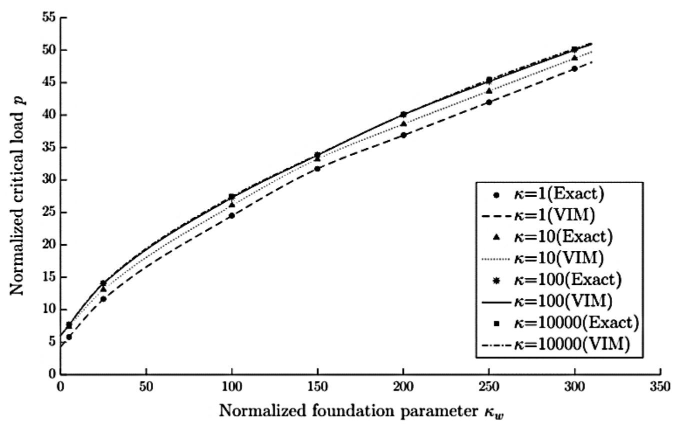


Figure 4. Buckling of column on elastic foundations ($\kappa_w/\kappa_p = 25$).

$$k_w = \frac{E_c}{6t_c} \qquad k_p = \frac{G_c t_c}{3} \tag{17}$$

$$\kappa_w = \frac{2}{\pi^4}\gamma\chi^3\eta_E \qquad \kappa_p = \frac{4}{\pi^2}\tau\chi^2\eta_G \tag{18}$$

where $\eta_E$ is the ratio of core elastic modulus to column elastic modulus $E_c/E$, $\eta_G$ is the ratio of core shear modulus to column elastic modulus $G_c/E$, $\gamma$ is the ratio of column length to core thickness $L/t_c$, $\tau$ is the ratio of core thickness to column thickness and $t_c/t$ and $\chi$ is the ratio of column length to thickness $L/t$. The equation shows the use of core material with higher elastic and shear modulus is advantageous. However, stiffer material is usually denser, which will increase the structure weight. $\kappa_w$ and $\kappa_p$ and proportional to the cubic and square of length to thickness ratio, respectively, so for a constant column thickness, increasing the column length (web-pitch) leads to higher buckling loads.

The typical web-pitch length used in marine industries is 120 mm and the core thickness is 40 mm. To explicitly demonstrate the effect of geometric parameters, different pitch lengths and core thicknesses are evaluated. The pitch lengths include 120 mm, 160 mm and 200 mm and the core thickness include 30 mm, 40 mm and 50 mm. The thickness of the column remains 2 mm. In practice, the filling foam material can be soft and light or rigid and dense. Various foams can be chosen according to the requirements of different applications. Four foams with different densities and mechanical properties are selected, which include rigid Polyurethane foam and Divinycell H-grade foam H45, H100 and H250. The property parameters of the foams are listed in Table 4. The normalized foundation parameters corresponding to different foam and geometry combinations are obtained, Table 5.

In table 5, it can be seen that for the same pitch length, $\kappa_w$ decrease with the increase of core thickness while $\kappa_p$ increase with the increase of core thickness for all the foams. For the same core thickness, $\kappa_w$becomes dramatically large for long pitch and $\kappa_p$also increase apparently. The results are identical with the predictions by Equation 18.

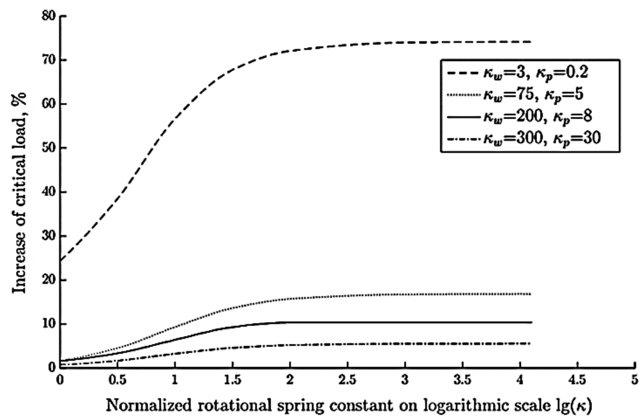


Figure 5. Illustration of the effect of rotational spring constants.

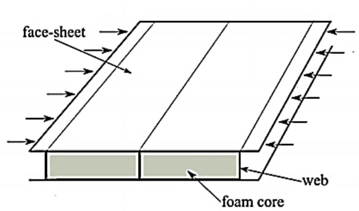


Figure 6. Illustrate of web core panel.

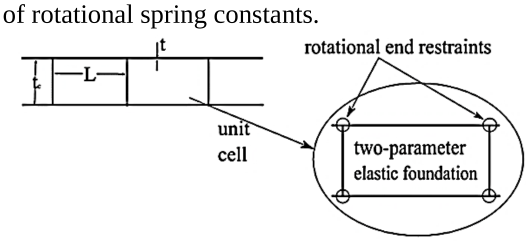


Figure 7. unit cell web core sandwich panel.

Based on the normalized foundation parameters, the critical buckling load of column with clamped- clamped boundary conditions ($\kappa = 10^9$) is evaluated, Table **6**. For the same pitch length, the critical loads are close for Rigid PU, H45 and H100 although the thickness are different, which means the effect of panel thickness is insignificant in this range. The thickness has larger influences with the application of the H250. This finding is desired for engineering design when the soft core is used because the smaller thickness can be used to reduce the structural weight. It is also found that for the same core thickness, the pitch length has significant effect. The effect of foams properties are also evident. Some values concerning beam buckling on soft and stiff core with SS and CC boundary conditions are listed in Table 6. For some conditions such as long pitch and stiffer core, the effect of boundary conditions can be ignored due to their trivial influence. For example, when the web-pitch of 200 mm and the core thickness of 30 mm and H250 are used as core material, the critical load with pin-pin boundary conditions is 52.1120 while that of clamped-clamped boundary conditions is 54.8823. The different is less than 5%. The rotational restraints can be treated as simply supported and the results will not be too conservative. However, when short pitch and soft core is used, the rotational boundary condition should be considered. The stiffness the webs provided should be evaluated for the latter case. Within the restrictions of other design criteria, longer pitch should be used to reduce the number of web used for a given total width of panel to reduce the structure weight.

| property | PU [25] | H45 [33] | H100 [33] | H250 [33] |
|---|---|---|---|---|
| density [kg/m$^3$] | 32 | 45 | 100 | 250 |
| elastic modulus [MPa] | 5.17 | 45 | 115 | 240 |
| shear modulus [MPa] | 1.58 | 12 | 28 | 88 |

Table 4. Foam core properties

| $t_c$ $[mm]$ | | 30 | | | 40 | | | 50 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $L$ $[mm]$ | | 120 | 160 | 200 | 120 | 160 | 200 | 120 | 160 | 200 |
| Rigid PU | $\kappa_w$ | 0.4474 | 1.4140 | 3.4520 | 0.3355 | 1.0605 | 2.5890 | 0.2684 | 0.8484 | 2.0712 |
| | $\kappa_p$ | 0.1687 | 0.2999 | 0.4685 | 0.2249 | 0.3998 | 0.6247 | 0.2811 | 0.4998 | 0.7809 |
| H45 | $\kappa_w$ | 3.8941 | 12.3072 | 30.0468 | 2.9205 | 9.2304 | 22.5351 | 2.3364 | 7.3843 | 18.0281 |
| | $\kappa_p$ | 1.2811 | 2.2775 | 3.5586 | 1.7081 | 3.0367 | 4.7448 | 2.1352 | 3.7958 | 5.9310 |
| H100 | $\kappa_w$ | 9.9515 | 31.4516 | 76.7862 | 7.4636 | 23.5887 | 57.5897 | 5.9709 | 18.8710 | 46.0717 |
| | $\kappa_p$ | 2.9892 | 5.3142 | 8.3034 | 3.9856 | 7.0856 | 11.0712 | 4.9820 | 8.8570 | 13.8390 |
| H250 | $\kappa_w$ | 20.7683 | 65.6382 | 160.2495 | 15.5762 | 49.2286 | 120.1871 | 12.4610 | 39.3829 | 96.1497 |
| | $\kappa_p$ | 9.3947 | 16.7017 | 26.0964 | 12.5263 | 22.2689 | 34.7952 | 15.6578 | 27.8361 | 43.4940 |

Table 5. Normalized foundation parameters for different foams and geometries

| $t_c$ $[mm]$ | | 30 | | | 40 | | | 50 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $L$ $[mm]$ | | 120 | 160 | 200 | 120 | 160 | 200 | 120 | 160 | 200 |
| Rigid PU | $p$(VIM) | 4.5029 | 5.3466 | 6.9685 | 4.4758 | 5.1875 | 6.5180 | 4.4819 | 5.1312 | 6.3039 |
| | $p$(Exact) | 4.5029 | 5.3466 | 6.9685 | 4.4758 | 5.1875 | 6.5180 | 4.4819 | 5.1312 | 6.3039 |
| H45 | $p$(VIM) | 8.0864 | 12.9365 | 17.6543 | 7.8360 | 13.0828 | 17.4072 | 7.8484 | 12.8592 | 17.7174 |
| | $p$(Exact) | 8.0864 | 12.9365 | 17.6543 | 7.8360 | 13.0828 | 17.4072 | 7.8484 | 12.8592 | 17.7174 |
| H100 | $p$(VIM) | 13.1793 | 19.6739 | 29.5572 | 13.0967 | 19.9512 | 30.0451 | 13.1663 | 20.8081 | 30.8564 |
| | $p$(Exact) | 13.1793 | 19.6739 | 29.5572 | 130967 | 19.9512 | 30.0451 | 13.1663 | 20.8081 | 30.8564 |
| H250 | $p$(VIM) | 21.7151 | 36.8650 | 54.8823 | 23.8315 | 39.8359 | 60.0767 | 26.3473 | 43.6591 | 66.5791 |
| | $p$(Exact) | 21.7151 | 36.8650 | 54.8823 | 23.8315 | 39.8359 | 60.0767 | 26.3473 | 43.6591 | 66.5791 |

Table 6. Normalized critical loads for different foams and geometries

| | | $t_c = 30, L = 200$ | $t_c = 40, L = 160$ | $t_c = 50, L = 120$ |
|---|---|---|---|---|
| Rigid PU(5.17,1.58) | CC | 6.9685 | 5.1875 | 4.4819 |
| | SS | 4.9205 | 2.4603 | 1.5495 |
| H250 | CC | 54.8823 | 39.8359 | 26.3473 |
| | SS | 52.1120 | 36.7387 | 22.7730 |

Table 7. Example of buckling load with CC and SS boundary conditions

# 6 CONCLUSION

Two analytical methods are presented in the paper to obtain the buckling solutions of columns resting on Pasternak foundations with rotational end restraints. Solutions from the two methods validate each other. Variational iteration method is used for this problem for the first time and is found convenient, efficient and accurate. The effects of rotational end restraints and Pasternak foundation parameters are investigated simultaneously. Rotational end restraints significantly increase critical buckling loads when weak foundations (foams) are used and the effects are weakened when denser and stronger foundations (foams) exists. Geometric and material properties are incorporated in the normalized foundation parameters. Practical structural geometries and foam materials are used to further evaluate foundation parameters. For some cases where long web-pitch and stiffer foams are used, the rotationally constrained boundary conditions can be simplified as simply supported without giving simplistic results.

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