Fast and Accurate Analytical Solution for Predicting Local Bending in Composite Conical Shell Structures with Flanges

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1. Abstract

Composite conical shells are widely used in aerospace structures as load-carrying components such as payload adapters, interstages, and intertanks. Although membrane theories accurately describe their global behaviour, they cannot capture the localized bending effects arising at shell–flange junctions, where the highest stress levels often occur. This work presents a semi-analytical formulation for the rapid prediction of local bending effects in flanged composite conical shells. The methodology combines the bending theory of laminated composite conical shells with analytical solutions for composite annular flanges, while the governing shell equations are solved using a fourth-order finite difference scheme. Compatibility conditions at the shell–flange interface are enforced to determine the localized discontinuity loads. The proposed approach is validated against refined finite element analyses, showing excellent agreement in terms of displacement and stress-resultant distributions. Owing to its high accuracy and low computational cost, the method represents an effective design-oriented tool for the conceptual and preliminary design of aerospace structures incorporating composite conical shells.

2. Introduction

The pace at which the aerospace industry is required to deliver new products is continuously increasing, driven by the steady growth of research programmes and companies in the field, as well as by increasingly intense competition. Consequently, the duration of the design phase must be kept under strict control, particularly during the conceptual design stage, where numerous alternative configurations must be compared and structurally assessed to support strategic decisions and identify the solution to be finalised and optimised. Although numerical methods, such as the finite element method, have nowadays become highly reliable and widely adopted in engineering design, analytical methods continue to demonstrate an unparalleled combination of swiftness and accuracy.

In this context, composite conical shells represent one of the most widespread geometries in aerospace structures, being commonly employed in launchers and spacecraft as load-bearing components, for example as payload adapters, interstages, or intertanks. While the overall structural behaviour of composite conical shells can be suitably evaluated using membrane theory, the regions connecting the shell to the flanges fall outside the scope of membrane modelling due to the presence of geometric discontinuities. Moreover, these regions are characterised by local bending effects arising from the need to ensure compatibility of the generalised displacements between the conical shell and the flanges. Although the resulting local stress state decays over a very short distance from the discontinuity, it typically gives rise to the highest stress levels within the structure. Therefore, a rigorous and accurate estimation of these local bending effects is of paramount importance to support the selection of the laminate lay-up and thickness of the composite conical shell.

Although composite shell structures have been extensively studied, analytical formulations specifically addressing localized edge disturbances and discontinuity-induced bending effects remain limited. Early work by Tutuncu [1] analysed balanced-symmetric laminated spherical shells subjected to edge loads using classical shell theory. Through infinite-series solutions, the study showed that laminate anisotropy strongly affects stress redistribution and localized edge effects.

A significant advancement was later introduced by Zingoni [2], who investigated junction effects in connected isotropic conical shells through asymptotic formulations based on the Geckeler simplifying assumption. The method expresses the shell response in terms of auxiliary variables, such as meridional rotations and transverse shear forces, and represents the structural behaviour as the superposition of membrane and localized bending solutions required to enforce compatibility at shell junctions. This analytical framework was later extended in [3] to non-shallow polynomial shells of revolution with strongly varying curvature.

Similar concepts were subsequently applied to composite pressure vessels. Belardi et al. [4] developed a closed-form bending theory based on the Donnell-Mushtari-Vlasov shell formulation, demonstrating that dome-cylinder transitions generate localized bending moments and shear forces that can produce stresses significantly higher than membrane contributions. The formulation showed excellent agreement with numerical simulations while maintaining low computational cost.

Despite the extensive literature on composite conical shells, localized bending effects in flanged configurations remain largely unexplored, and no general analytical framework is currently available for coupled shell-flange analysis under edge loads.

Accordingly, this work aims to provide a design-oriented tool based on a semi-analytical solution of the bending theory for composite conical shells, capable of assessing the local stresses induced by the flanges connecting the shell to adjacent structures. The proposed methodology originates from the formulation of the equilibrium and compatibility equations for a composite conical shell, leading to a governing system of differential equations solved using the finite difference method with fourth-order accuracy. This solution, combined with that of a composite circular plate used to model the flange behaviour, enables an accurate evaluation of the local bending effects in the conical structure in terms of both stress and displacement fields.

The results are successfully validated through comparison with finite element simulations, demonstrating the accuracy and reliability of the proposed methodology.

3. Theoretical framework

A composite conical shell with flanged terminations is formed by a truncated conical segment connected to annular flanges at both ends, as illustrated in Fig. 1, where the reference mid-surface configuration is depicted.

The geometric parameters of the conical portion (see Fig. 1) include the radii at the two ends, indicated by R_1 and R_2, the axial height H, the slant length L, and the semi-apex angle α. The flanges are connected to the shell at the corresponding inner radii, whereas their external radii are identified as R_1^o and R_2^o for the upper and lower flange, respectively.

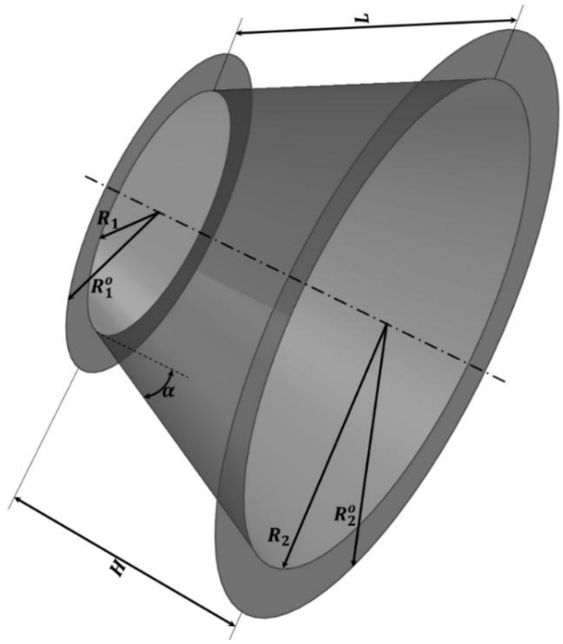

Figure 1. Composite conical shell with flanged terminations.

The entire structure is assumed to be manufactured from laminated composites composed of N fibre-reinforced orthotropic plies arranged according to a symmetric lay-up, leading to an overall thickness equal to t. Hence, corresponding stress resultants-strain equations are:

$$
\begin{aligned}
N_x &= A_{11}\frac{du}{dr} + A_{12}\frac{u}{r} \\
N_\theta &= A_{12}\frac{du}{dr} + A_{22}\frac{u}{r} \\
M_x &= -D_{11}\frac{d^2w}{dx^2} - D_{12}\frac{1}{x}\frac{dw}{dx} \\
M_\theta &= -D_{12}\frac{d^2w}{dx^2} - D_{22}\frac{1}{x}\frac{dw}{dx}
\end{aligned}
\tag{1}
$$

Once the upper flange of the structure is vertically loaded with the force per unit length F_e applied at its outer radius, at the junction between the conical shell and the upper flange self-equilibrated and localised discontinuity loads arise, as shown in Fig.Figure 2. These loads are the in-plane force N_x^0, the shear force Q_x^0 and the bending moment M_x^0.

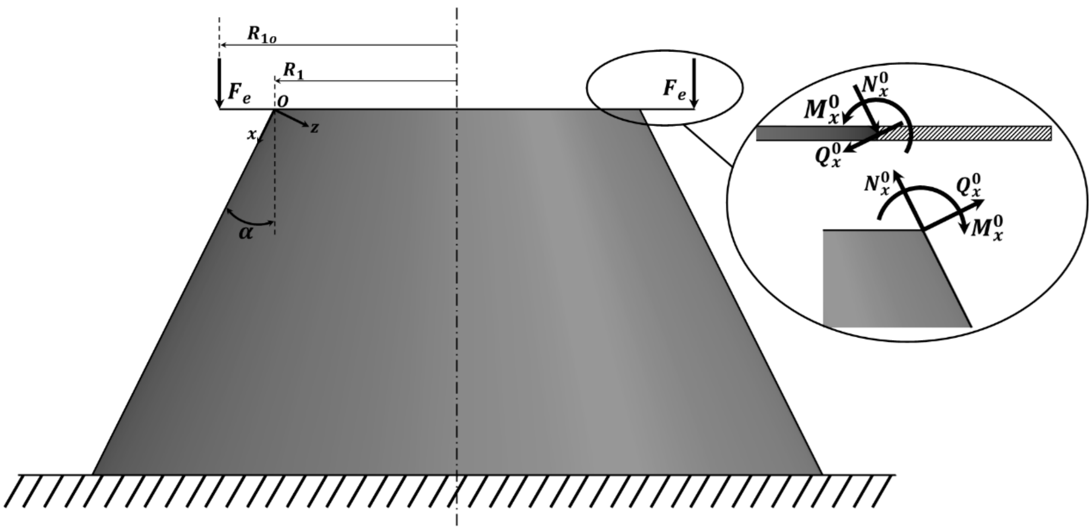

Figure 2. External vertical force per unit length F_e applied at the structure and localised discontinuity loads between the conical shell and the upper flange.

3.1. Conical shell

A local cylindrical coordinate system $O(x, \theta, z)$ is introduced to define the shell kinematics and stress resultants, as schematically reported in Fig.Figure 3. The origin of the system is positioned on the shell middle surface at the smaller-radius edge. The coordinate x follows the meridional direction along the shell generator, θ represents the circumferential coordinate, and z is directed through the thickness toward the axis of revolution. The kinematic description of the middle surface is completed by introducing the displacement fields u, v, and w, corresponding to the three directions of the adopted coordinate system.

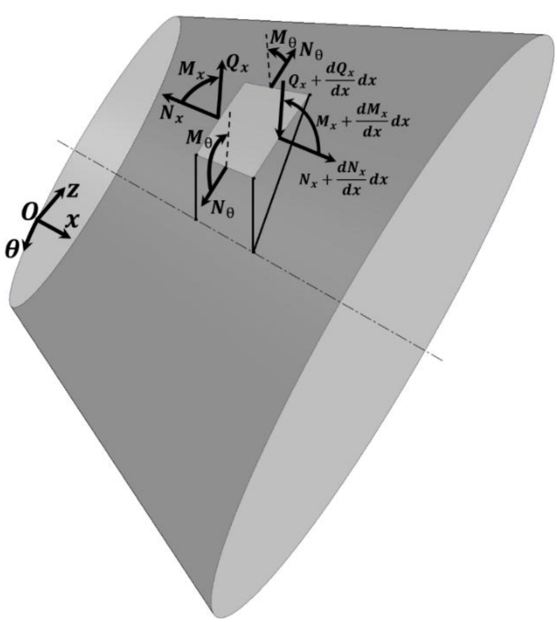

Figure 3. Differential element of composite conical shell.

Regarding the conical shell, the relations between the strain and curvature and the mid-surface displacements are:

$$\begin{aligned} \varepsilon_x^0 &= \frac{du}{dx} & \kappa_x &= -\frac{d^2w}{dx^2} \\ \varepsilon_\theta^0 &= \frac{u\sin\alpha - w\cos\alpha}{R(x)} & \kappa_\theta &= -\frac{\sin\alpha}{R(x)}\frac{dw}{dx} \\ \gamma_{x\theta}^0 &= 0 & \kappa_{x\theta} &= 0 \end{aligned} \tag{2}$$

After obtaining the equilibrium equations of the differential element of conical shell in Fig.Figure 3, and considering the constitutive equations Eq. (1) and the relations in Eq. (2), the two differential governing equations are obtained for the bending theory of composite conical shells:

$$A_{11}\frac{d^2u}{dx^2} + A_{11}\frac{\sin\alpha}{R(x)}\frac{du}{dx} - A_{12}\frac{\cos\alpha}{R(x)}\frac{dw}{dx} - A_{22}\sin\alpha\frac{u\sin\alpha - w\cos\alpha}{R^2(x)} = 0 \tag{3}$$

and

$$\begin{aligned} &D_{11}\frac{d^4w}{dx^4} + 2\frac{D_{11}\sin\alpha}{R(x)}\frac{d^3w}{dx^3} - \frac{D_{22}\sin^2\alpha}{R^2(x)}\frac{d^2w}{dx^2} - \frac{A_{12}\cos\alpha}{R(x)}\frac{du}{dx} + \frac{D_{22}\sin^3\alpha}{R^3(x)}\frac{dw}{dx} \\ &+ \frac{A_{22}\cos\alpha}{R^2(x)}(w\cos\alpha - u\sin\alpha) = 0 \end{aligned} \tag{4}$$

The unknown displacement components in the system of differential equations, Eqs. (3) and (4), are the meridional displacement u and the transverse displacement w; these displacement components describe the localised bending behaviour of the conical shell loaded with edge loads.

The set of boundary conditions to be imposed for the solution of Eqs. (3) and (4) are:

$$\begin{aligned} N_x(0) &= N_x^0 & u(L) &= 0 \\ Q_x(0) &= Q_x^0 & w(L) &= 0 \\ M_x(0) &= M_x^0 & \phi_x(L) &= 0 \end{aligned} \tag{5}$$

Eq. (5) implies the application of the edge loads that arise at the small radius R_1 (i.e. meridian coordinate $x = 0$) as an effect of the connection with the upper flange and the clamped condition at the large radius R_2 (i.e. meridian coordinate $x = L$).

Eqs. (3) and (4) do not allow to obtain a closed-form solution because of coupling of unknown functions and variable coefficients. Hence, a solution approach based on the Finite Difference Method is presented in [5].

3.2. Circular flange

Each flange is represented as an annular component and formulated within a cylindrical reference system, as illustrated in Fig.Figure 4.

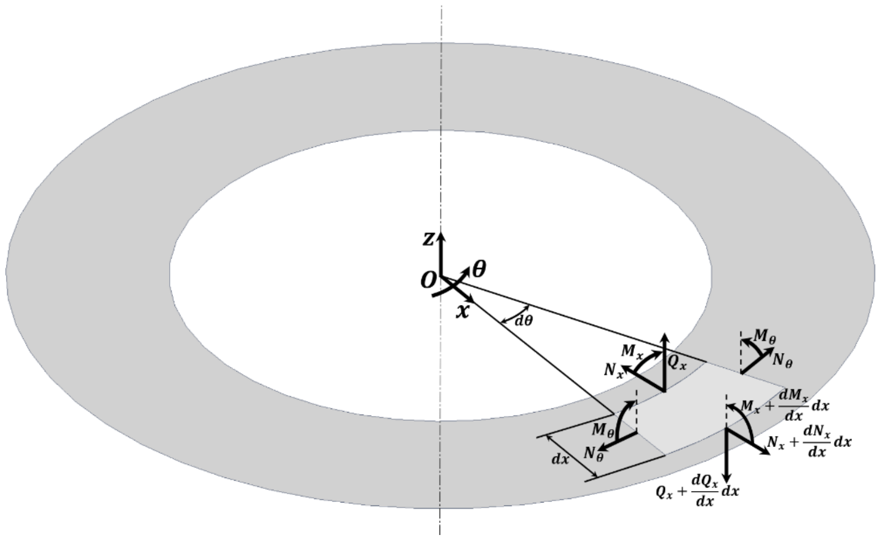

Figure 4. Differential element of composite flange.

The localised edge loads acting at the junction between the conical shell and the upper flange imply both the bending and the in-plane loading of the flange. Thus, this element must be analysed as a plate and as a disk.

In the first case, the deflection w of the upper flange considered as a composite plate is:

$$w(x) = C_0 + \frac{C_1}{\lambda_D + 1} x^{\lambda_D + 1} + \frac{C_2}{1 - \lambda_D} x^{1 - \lambda_D} - \frac{F_e R_{1o}}{2(D_{11} - D_{22})} x^2 \tag{6}$$

in which:

$$\lambda_D = \sqrt{\frac{D_{22}}{D_{11}}} \tag{1}$$

and the integration constants C_0, C_1 and C_2 are determined from the boundary conditions:

$$\begin{gathered} w(R_1) = 0 \\ M_x(R_1) = -M_x^0 \\ \phi_x(R_{1o}) = 0 \end{gathered} \tag{7}$$

Also, the rotation of the normal to the mid-surface of the plates is given by:

$$\phi_x = -\frac{dw}{dx} \tag{8}$$

Analogously, the radial displacement u of the upper flange when loaded as a disk is:

$$u(x) = K_1 x^{\lambda_A} + K_2 x^{-\lambda_A} \tag{9}$$

where:

$$\lambda_A = \sqrt{\frac{A_{22}}{A_{11}}} \tag{2}$$

The integration constants K_1 and K_2 are derived from the boundary conditions:

$$\begin{aligned} \sigma_x(R_1) &= \frac{Q_x^0 \cos\alpha - N_x^0 \sin\alpha}{t} \\ u(R_{1o}) &= 0 \end{aligned} \tag{10}$$

3.3. Localised discontinuity loads

The determination of the localised discontinuity loads N_x^0, Q_x^0 and M_x^0 is obtained requiring the continuity of the displacements and rotations at the section where the conical shell and the upper flange are connected. This implies that following conditions must be satisfied:

$$\begin{aligned} &\phi_x^f(R_1) = \phi_x^c(0) \\ &(N_x^0 \cos\alpha + Q_x^0 \sin\alpha) 2\pi R_1 = -F_e 2\pi R_{1o} \\ &u^f(R_1) = u^c(0)\sin\alpha + w^c(0)\cos\alpha \end{aligned} \tag{11}$$

stating, respectively, that rotations of the junction section are continuous across the flange and the conical shell, the external load is transferred to the conical shell, the radial displacement is continuous between the two components.

4. Results

The reported procedure is tested using a flanged composite conical shell with height $H = 1500$ mm, small radius of the conical shell $R_1 = 700$ mm, outer radius of the upper flange is $R_1^o = 1000$ mm, semi-apex angle $\alpha = 35°$ and stacking sequence $[90/30/-30/45/-45]_s$. The mechanical properties of the fibre-reinforced orthotropic layers are reported in Table 1.

Table 1. Mechanical properties.

Property	Value (GPa)	Property	Value (GPa)	Property	Value (GPa)
E_{11}	104	G_{12}	5.2	υ_{12}	0.3
E_{22}	10	G_{13}	5.2	υ_{13}	0.3
E_{33}	10	G_{23}	3.95	υ_{23}	0.5

The composite shell undergoes a vertical load per unit length F_e of unitary intensity as shown in Fig.Figure 2.

The results of displacement field and stress resultants reported in Fig.Figure 5 and Fig.Figure *6*, respectively, are obtained applying the procedure outlined in the previous sections. The diagrams compare the results of the semi-analytical procedure with those of refined Finite Element (FE) models developed featuring four-node shell elements.

The results are normalized with respect to largest value evaluated with the FE analysis. The abscissa is likewise normalized, beginning at the outer radius of the flange and progressing radially toward the conical shell before continuing along the shell meridian.

Overall, the agreement between the proposed bending theory and the finite element results is excellent, with only negligible discrepancies observed. Furthermore, the proposed approach accurately and efficiently predicts the peak values of the stress resultants, thereby supporting informed design decisions during the early stages of the development process without the need for detailed finite element modelling.

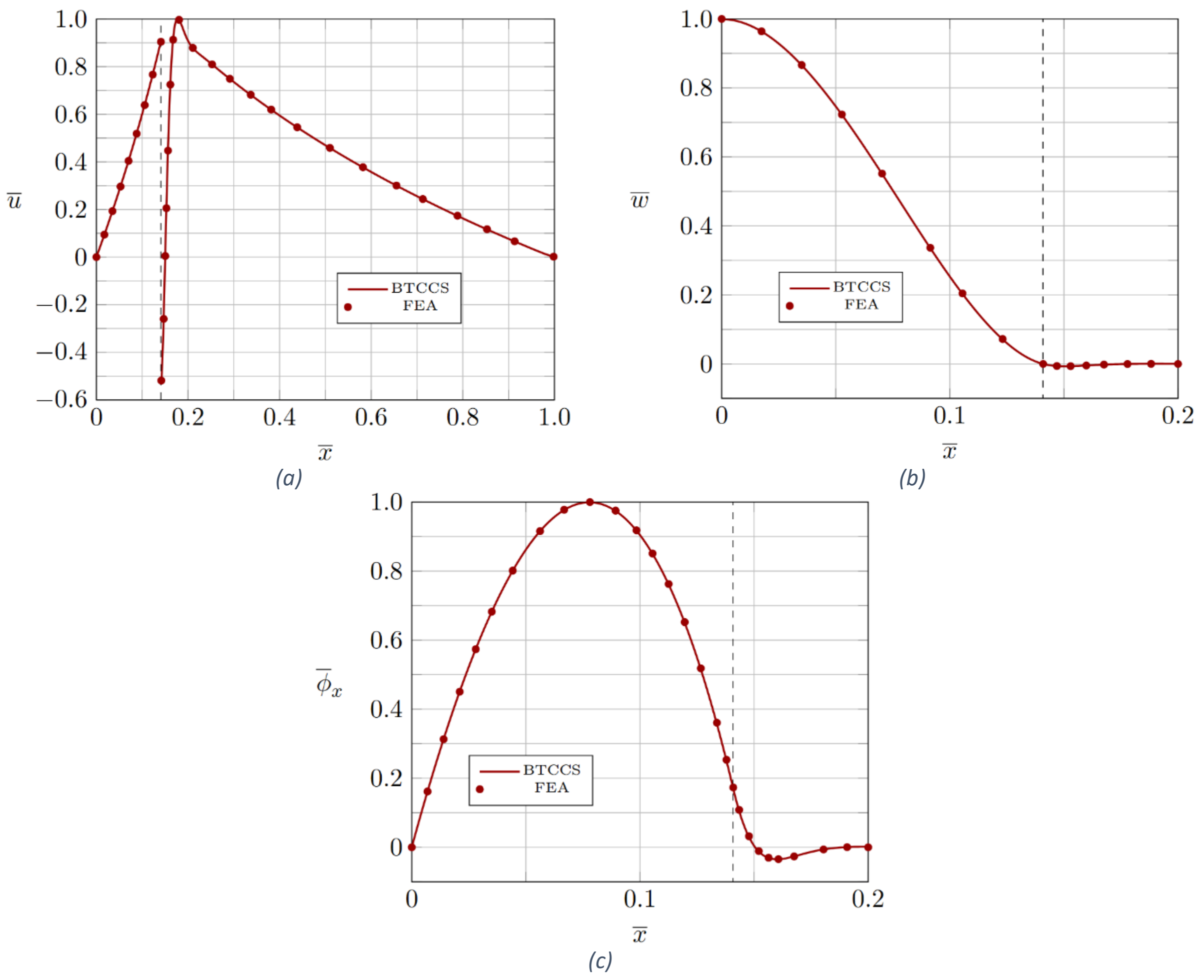

Figure 5. Flanged composite conical shell, normalized: (a) meridian displacement $\overline{u}$, (b) transversal displacement $\overline{w}$ and (c) rotation about the circumferential direction $\overline{\phi}_x$.

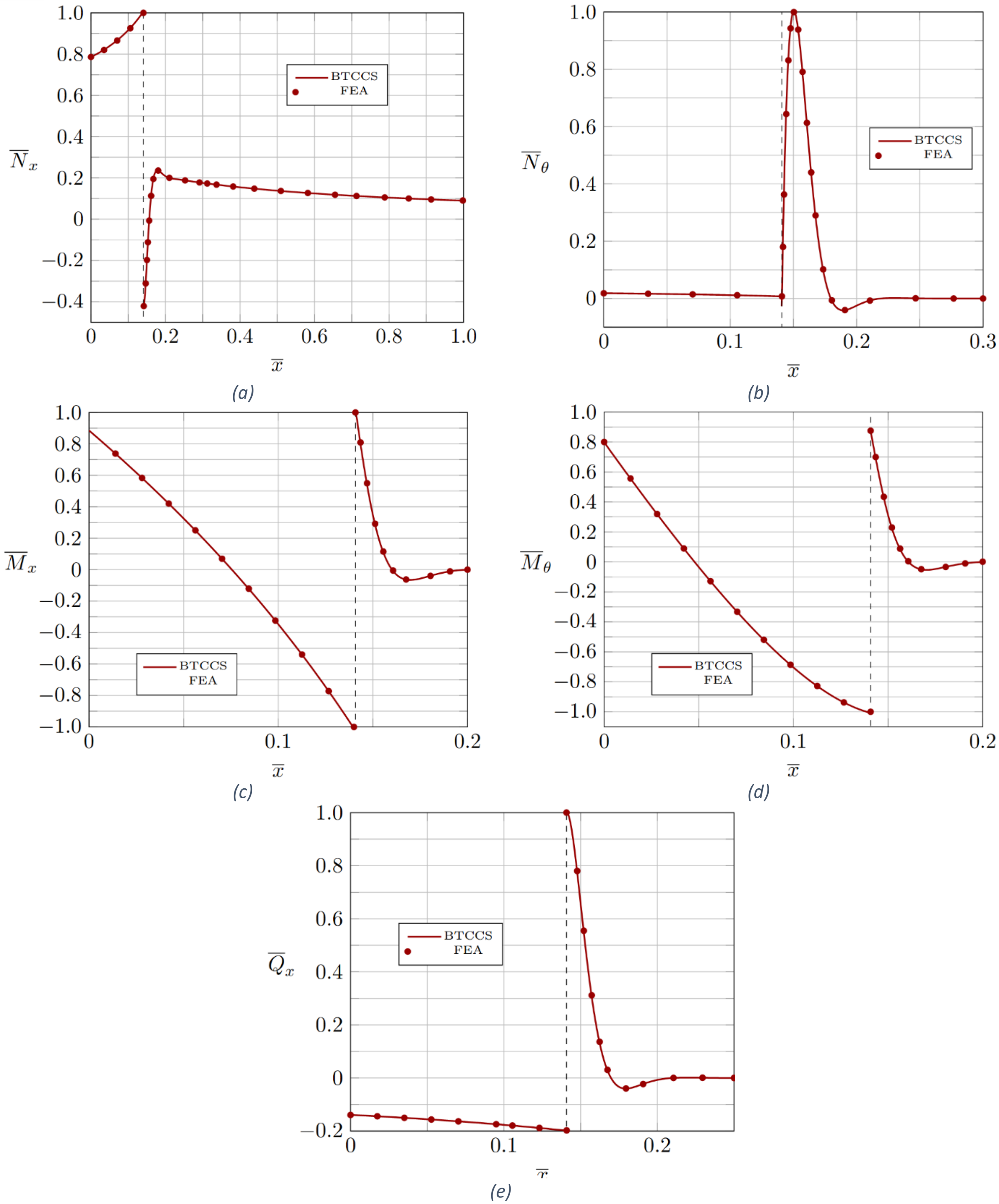

Figure 6. Flanged composite conical shell, normalized: (a) meridian stress resultant $\overline{N}_x$, (b) circumferential stress resultant $\overline{N}_\theta$, (c) meridian bending moment $\overline{M}_x$, (d) circumferential bending moment $\overline{M}_\theta$ and (e) shear force $\overline{Q}_x$.

5. Conclusions

A semi-analytical methodology for the evaluation of localized bending effects in flanged composite conical shells has been presented. The formulation combines the bending theory of laminated composite conical shells with analytical solutions for composite annular flanges, while the governing shell equations are solved through a fourth-order finite difference scheme. The interaction between the shell and the flange is represented through self-equilibrated discontinuity loads obtained by enforcing displacement and rotation compatibility at the junction.

The proposed approach was applied to a representative flanged composite conical shell subjected to edge loading and validated against refined finite element analyses. The comparison demonstrated excellent agreement in terms of displacement fields, stress resultants, bending moments, and transverse shear forces, confirming the capability of the formulation to accurately capture the localized structural response associated with shell–flange connections.

The results highlight that the proposed methodology provides a computationally efficient alternative to detailed finite element modelling while retaining a high level of accuracy. Consequently, it constitutes a valuable design tool for the conceptual and preliminary design of aerospace structures incorporating composite conical shells, enabling rapid evaluation of local stress concentrations and supporting informed decisions regarding laminate configuration and structural sizing.

6. References

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