Formulation of a temperature-dependent viscoelastic-damage-friction constitutive model to quantify entropy generated during fatigue of composite materials

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1. Abstract

The concept of entropy generation – commonly denoted as Fracture Fatigue Entropy (FFE) – has shown potential as a unifying fatigue failure criterion. Existing numerical applications have been limited to damage-coupled linear elastic constitutive models, with viscoelasticity and internal friction either modeled as a single phenomenon or disregarded. Moreover, there have been limited validation tests on thermoplastic composites, which show stronger rate and temperature dependencies than the thermoset counterparts. This work describes a viscoelastic-damage-friction constitutive model weakly coupled with temperature. The model formulated in this work is to be numerically implemented and validated against experimental data in future work.

2. Introduction

Research in fatigue of composite materials has been driven by the utopian goal of developing a unified model for fatigue life prediction of any stacking sequence subject to any loading condition. The high variability in mechanical behavior resulting from the intrinsic heterogeneity of composite materials has limited the predictability of existing fatigue models to specific case scenarios. Within analytical methods, one interesting alternative consists of using entropy generation as a fatigue failure criterion. The approach was formally introduced for fatigue life prediction in metals by Naderi et al. [1] and composites by Naderi and Khonsari [2] who renamed it to the now-common term FFE. However, existing fatigue life prediction models implementing FFE as failure criterion [3] have three main shortcomings: 1) constitutive models are assumed linear elastic, with viscoelasticity considered as additional source of dissipation, 2) friction effects are disregarded, 3) experimental tests on FFE of thermoplastics are limited and lack computational reproducibility. In this work, an analytical mesoscale constitutive model valid in the small deformation regime is proposed. Linear viscoelasticity is modeled using the Boltzmann superposition integral with a time-dependent stiffness tensor represented as a Prony series. The Continuum Damage Mechanics (CDM) approach is used to introduce a second-order damage tensor in the undamaged viscoelastic formulation, plus a damage-friction coupling term acting on the damaged portion of the material.

Thermal and mechanical effects are weakly coupled through the first law of thermodynamics, thus allowing temperature to change on a cycle-by-cycle basis and affect the temperature-dependent model parameters. This document is outlined as follows: the 1D rheological model is first introduced as motivation to a generalized 3D model; the constitutive model is then conceptually formulated, followed by a detailed description of the linear viscoelastic and damage models; the challenges of coupling damage-friction-viscoelastic models are presented, concluded by the evolution laws for damage, sliding friction and temperature variables.

3. Theoretical background

3.1. One-dimensional rheological model

We first introduce the rheological model that describes the 1D material behavior in the presence of crack sliding as baseline for a more general 3D model. The model shown in Figure 1 is an extension of the model first proposed by Ragueneau et al. [4] by adding Maxwell branches parallel to the Hooke solids to simulate viscous relaxation.

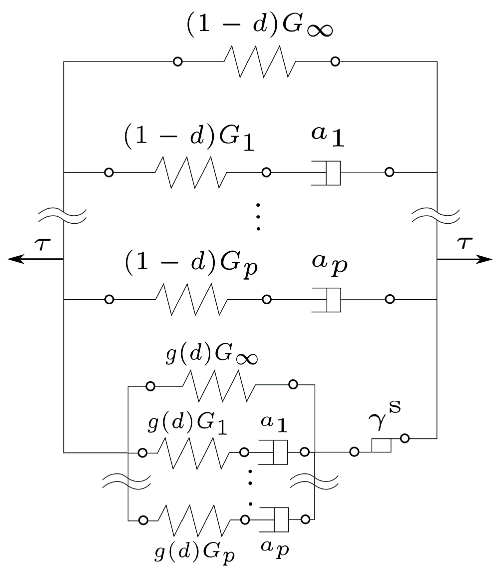

Figure 1. 1D rheological model of the material behavior coupling linear viscoelasticity, damage and friction under shear stress.

In the presence of a scalar damage variable, d, the total shear stress in the material results from viscoelastic and internal friction effects, τ^{ve} and τ^s, respectively,

$$\tau = (1-d)\tau^{ve} + g(d)\tau^s. \tag{1}$$

The viscoelastic term acts on the undamaged (or continuous) portion of the material, while the internal friction stress component is due to the relative sliding of two crack surfaces on the damaged (or discontinuous) part. If no relative sliding occurs, it is assumed that bonded cracks deform under the same viscoelastic properties as the undamaged counterpart. The coupling between damage and internal friction must be captured through the function $g(d)$. When the most critical failure condition is attained (i.e. $d = d_c = 1$), the material behavior should be such that only frictional sliding occurs. Moreover, experimental evidence in cracked brittle materials shows a significant restoration of shear stiffness because of the blockage effect caused by frictional sliding [5]. One can intuitively assume the blockage effect to intensify as damage evolves. Therefore, $g(d)$ should be a monotonic increasing function, with the simplest linear form, $g(d) = d$ considered in [4,6].

3.2. Dissipation inequality

In this work, the importance of the second law of thermodynamics quantified by the Clausius-Duhem inequality is twofold: to guarantee that the proposed constitutive law is physically admissible, and quantify the entropy generated due to dissipative phenomena during fatigue. The latter is our ultimate goal with the expectation that a unique value emerges as a potential failure criterion; the former is the focus of this section.

In the realm of infinitesimal strains, the second law of thermodynamics for a general non-isothermal case can be described with respect to a single frame of reference as follows,

$$\dot{\gamma}_{gen} = \frac{1}{\theta}\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}} - \frac{\rho}{\theta}\left(\dot{\psi} + \dot{\theta}\eta\right) - \boldsymbol{q}\cdot\frac{\nabla\theta}{\theta^2} \geq 0, \tag{2}$$

where γ_{gen} is the generated entropy density, $\boldsymbol{\sigma}$ is the small deformation stress tensor, $\boldsymbol{\varepsilon}$ is the small deformation total strain tensor (assumed viscoelastic), ρ is the density of the continuum, ψ is the specific Helmholtz free energy, θ is the absolute temperature of the continuum, η is the specific internal entropy, and $\boldsymbol{q}$ is the heat transfer rate. The terms *density* and *specific* used in the definitions of energies and entropy refer to quantities per unit volume and per unit mass, respectively.

The material behavior must be quantified through one of the thermodynamic potentials in equation (2). In this work, carbon fibre-reinforced polymers (CFRP) are modeled as viscoelastic materials coupled with damage, internal friction and temperature, quantified through ψ. (Visco)plastic effects were not accounted for to simplify the model construction. The simplest form of ψ is a function of the following variables,

$$\psi = \psi(\boldsymbol{\varepsilon}, \boldsymbol{D}, \boldsymbol{\varepsilon}^s; \theta), \tag{3}$$

where $\boldsymbol{\varepsilon}$ is an observable variable, θ is treated as a parameter, and the internal tensor variables and $\boldsymbol{D}$ and $\boldsymbol{\varepsilon}^s$ are the damage and sliding strain, respectively. The Helmholtz free energy can be further split into viscoelastic and internal friction contributions, ψ^{ve} and ψ^s, respectively,

$$\psi = \psi^{ve}(\boldsymbol{\varepsilon}, \boldsymbol{D}; \theta) + \psi^s(\boldsymbol{\varepsilon}, \boldsymbol{D}, \boldsymbol{\varepsilon}^s; \theta), \tag{4}$$

$$\psi^{ve} = \frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t} \frac{d\boldsymbol{\varepsilon}(\zeta)}{d\zeta} : \boldsymbol{C}^{ve}(t-\zeta, t-\tau) : \frac{d\boldsymbol{\varepsilon}(\tau)}{d\tau} d\tau d\zeta, \tag{5}$$

$$\psi^{s} = \frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t} \left(\frac{d\boldsymbol{\varepsilon}(\zeta)}{d\zeta} - \boldsymbol{\varepsilon}^s(t)\right) : \boldsymbol{C}^{f}(t-\zeta, t-\tau) : \left(\frac{d\boldsymbol{\varepsilon}(\tau)}{d\tau} - \boldsymbol{\varepsilon}^s(t)\right) d\tau d\zeta\,, \tag{6}$$

where $\boldsymbol{C}^{ve}$ and $\boldsymbol{C}^{f}$ are symmetric fourth-order damage-dependent stiffness tensors associated to the viscoelastic (undamaged) and sliding (damaged) domains, respectively. The form of these stiffness tensors will be detailed in the next sections.

3.3. Linear viscoelastic model

The linear viscoelastic behavior is modeled using the Boltzmann superposition principle quantified through the following convolution integral [7],

$$\boldsymbol{\sigma}^{ve} = \int_{-\infty}^{t} \boldsymbol{C}^{ve}(t-\tau) : \frac{d\boldsymbol{\varepsilon}(\tau)}{d\tau}\, d\tau, \tag{7}$$

where $\boldsymbol{C}^{ve}$ is represented in the form of a Prony series of p terms \cite{schapery1967},

$$\boldsymbol{C}^{ve}(t) = \boldsymbol{C}_{\infty} + \sum_{i=1}^{p} \boldsymbol{C}_{i}\, e^{-a_i t}, \tag{8}$$

where $a_i, i = 1, \dots, p$ are time constants.

3.4. Damage model

The framework of CDM first proposed by Kachanov [8] is here considered to incorporate the effect of damage in the constitutive law of the material. The geometric and mathematical descriptions of damage proposed by Talreja [9], Kachanov [8] and Murakami [10] are first introduced before presenting our proposed modification.

Assuming a pair of flat-like crack surfaces, i, with normal vector, $\boldsymbol{n}^{(i)}$, damage can be locally defined as,

$$\boldsymbol{d}^{(i)} = \int \boldsymbol{a}^{(i)} \otimes \boldsymbol{n}^{(i)}\, dS^{(i)}, \tag{9}$$

where $\boldsymbol{a}^{(i)}$ is the relative displacement between the pair of crack surfaces i, and $\boldsymbol{n}^{(i)}$ is the unit vector normal to the surface geometry when the crack is closed. The contribution from all locally formed cracks, p, is represented by the orthotropic damage tensor, $\boldsymbol{D}$ [11],

$$\boldsymbol{D} = \frac{1}{V} \sum_{i=1}^{p} \boldsymbol{d}^{(i)}, \qquad D = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{21} & D_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{10}$$

where V is a reference volume such that $\boldsymbol{D}$ is nondimensional. The drawback of equation (9) is that $\boldsymbol{D} = \boldsymbol{0}$ in the presence of cracks with no relative displacement. Kachanov [8] suggested an alternative definition where $\boldsymbol{a}$ is replaced with $\boldsymbol{n}$ in the definition of equation (9). In this work, we propose a refined definition of the damage tensor,

$$\boldsymbol{d}^{(i)} = \int \boldsymbol{n}^{(i)} \otimes \left(\boldsymbol{m}^{(i)} + \boldsymbol{n}^{(i)}\right) dS^{(i)}, \tag{11}$$

where $\boldsymbol{m}^{(i)}$ is a unit vector perpendicular to $\boldsymbol{n}^{(i)}$ in the 1-2 plane (i.e. $\boldsymbol{m}^{(i)} \cdot \boldsymbol{n}^{(i)} = 0$) which represents a direction of influence of the crack.

The physical interpretation of each term of $\boldsymbol{D}$ is schematically shown in Figure 2. In a general case, D_{12} and D_{21} are not necessarily equal, thus yielding an asymmetric damage tensor [12]. It is common practice to assume the damage tensor is symmetric, such as in [11,13]. Talreja [12] suggested splitting the asymmetric contribution to the damage tensor due to sliding between crack surfaces from the normal separation and disregard the contribution of the former.

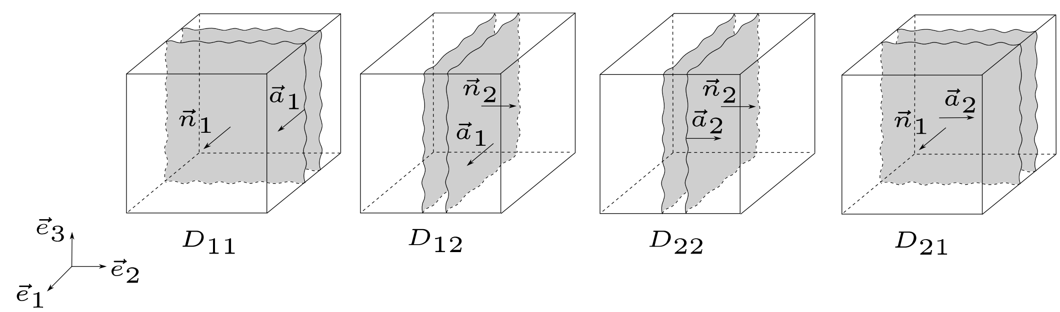

Figure 2. Damage modes.

Since no out-of-plane deformations are considered, and mesoscale models do not account for interlaminar damage modes, the terms $D_{i3}, i = 1,2,3$ are also zero. The damage tensor can be split into its symmetric ($\boldsymbol{D}^{sym}$) and skew-symmetric ($\boldsymbol{D}^{skew}$) parts as,

$$\boldsymbol{D} = \boldsymbol{D}^{sym} + \boldsymbol{D}^{skew}, \tag{12}$$

where,

$$\boldsymbol{D}^{sym} = \frac{1}{2}(\boldsymbol{D} + \boldsymbol{D}^T) = \begin{bmatrix} D_{11} & D_{12}^{sym} & 0 \\ D_{12}^{sym} & D_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_{12}^{sym} = \frac{D_{12} + D_{21}}{2}, \tag{13}$$

$$\boldsymbol{D}^{skew} = \frac{1}{2}(\boldsymbol{D} - \boldsymbol{D}^T) = \begin{bmatrix} 0 & \dfrac{D_{12} - D_{21}}{2} & 0 \\ \dfrac{D_{12} - D_{21}}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{14}$$

In this work, equation (1) is extrapolated to a 3D case as follows,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{ve} + \boldsymbol{\sigma}^{s} = \boldsymbol{M}^{-1}(\boldsymbol{D}^{*}){:}\,\overline{\boldsymbol{\sigma}}^{ve} + \boldsymbol{N}(\boldsymbol{D}){:}\,\boldsymbol{\sigma}^{f}, \tag{15}$$

where $\boldsymbol{M}$ and $\boldsymbol{N}$ are fourth-order linear transformation tensors, $\overline{\boldsymbol{\sigma}}^{ve}$ is the viscoelastic stress tensor in the undamaged configuration, and $\boldsymbol{\sigma}^{f}$ is the friction stress. The following sections detail the coupling of damage with viscoelasticity and friction.

3.5. Damage-viscoelastic coupling

To introduce the effect of damage in strain-space-formulated constitutive laws, it is convenient to use the notions of effective stress and the principle of equivalent strain. The simplest transformation of stress between the real and fictitious domains could be given by,

$$\boldsymbol{\sigma}^{ve} = (\boldsymbol{I} - \boldsymbol{D}^*)\overline{\boldsymbol{\sigma}}^{ve}, \tag{16}$$

where $\boldsymbol{D}^*$ is a second-rank damage tensor which can be distinct from $\boldsymbol{D}$ defined in equation (10). Regardless of the symmetry nature of $\boldsymbol{D}^*$, equation (16) yields $\boldsymbol{\sigma}^{ve}$ to be asymmetric which is not suitable for defining a constitutive law. Here, we present one possible approach to symmetrize $\boldsymbol{\sigma}^{ve}$, but the interested reader should refer to [10] for more alternatives. The symmetric part of the stress tensor defined in the real domain, $(\boldsymbol{\sigma}^{ve})^{\boldsymbol{s}}$ can be obtained as follows,

$$(\boldsymbol{\sigma}^{ve})^{\boldsymbol{s}} = \frac{1}{2}[\overline{\boldsymbol{\sigma}}^{ve}(\boldsymbol{I} - \boldsymbol{D}^*) + (\boldsymbol{I} - \boldsymbol{D}^*)\overline{\boldsymbol{\sigma}}^{ve}] = \boldsymbol{M}^{-1}(\boldsymbol{D}^*): \overline{\boldsymbol{\sigma}}^{ve}. \tag{17}$$

The constitutive law of equation (7) can be expressed in the fictitious undamaged configuration as,

$$\overline{\boldsymbol{\sigma}}^{ve} = \int_{-\infty}^{t} \boldsymbol{C_0}(t-\tau): \frac{d\overline{\boldsymbol{\varepsilon}}(\tau)}{d\tau} d\tau. \tag{18}$$

Using the strain equivalence hypothesis, $\overline{\boldsymbol{\varepsilon}} = (\boldsymbol{\varepsilon}\,(\boldsymbol{\sigma}^{ve})^{\boldsymbol{s}}, \boldsymbol{D})$ [14], equation (18) can be rewritten as,

$$\overline{\boldsymbol{\sigma}}^{ve} = \int_{-\infty}^{t} \boldsymbol{C_0}(t-\tau): \frac{d\boldsymbol{\varepsilon}(\tau)}{d\tau} d\tau. \tag{19}$$

Combining equations (17) and (19), and assuming $\boldsymbol{D}^*$ is only a function of the current time (i.e. independent of past history), the stress-strain constitutive law in the damaged configuration yields,

$$(\boldsymbol{\sigma}^{ve})^{\boldsymbol{s}} = \int_{-\infty}^{t} \boldsymbol{C_D}(t-\tau): \frac{d\boldsymbol{\varepsilon}(\tau)}{d\tau} d\tau, \qquad \boldsymbol{C_D}(\boldsymbol{D}^*) = \boldsymbol{M}^{-1}(\boldsymbol{D}^*): \boldsymbol{C_0}. \tag{20}$$

An undesired consequence of the principle of strain equivalence is the general asymmetry of $\boldsymbol{C_D}$. Since the damage tensor is a state variable of the system, an alternative definition of the stiffness tensor using the same symmetrization scheme implemented in equation (17) is considered,

$$\boldsymbol{C_D}(\boldsymbol{D}) = \frac{1}{2}[\boldsymbol{M}^{-1}(\boldsymbol{D}): \boldsymbol{C_0} + \boldsymbol{C_0}: \boldsymbol{M}^{-T}(\boldsymbol{D})]. \tag{21}$$

One can impose $\boldsymbol{C_D}$ to be always symmetric by modifying the state variable $\boldsymbol{D}^*$ such that $\boldsymbol{C_D}(\boldsymbol{D}) = \boldsymbol{C_D}(\boldsymbol{D}^*)$ is satisfied. Therefore, the relationship between $\boldsymbol{D}$ and $\boldsymbol{D}^*$ must be as follows [10,15],

$$\boldsymbol{M}^{-1}(\boldsymbol{D}^*) = \frac{1}{2}\left[\boldsymbol{M}^{-1}(\boldsymbol{D}) + \boldsymbol{C_0}: \boldsymbol{M}^{-T}(\boldsymbol{D}): \boldsymbol{C_0}^{-1}\right]. \tag{22}$$

In practice, $\boldsymbol{D}$ is directly quantified through experiments and equation (22) to obtain $\boldsymbol{D}^*$ will not be necessary.

3.6. Friction model

The friction stress tensor, $\boldsymbol{\sigma}^{\boldsymbol{f}}$, for a plane stress case can be described as the sum of normal and shear components,

$$\boldsymbol{\sigma}^f = (\boldsymbol{\sigma}^f)^n + (\boldsymbol{\sigma}^f)^t = \begin{bmatrix} \sigma_{11}^f & 0 & 0 \\ 0 & \sigma_{22}^f & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \sigma_{12}^f & 0 \\ \sigma_{12}^f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{23}$$

On the system of axes aligned with the crack direction, $\boldsymbol{\sigma}^f$ can be expressed as,

$$\boldsymbol{\sigma}^f = \begin{bmatrix} 1-h(\varepsilon_{nn}) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-h(\varepsilon_{nn}) \end{bmatrix} \begin{bmatrix} \sigma_{nn} \\ 0 \\ \sigma_{nt} \end{bmatrix}, \tag{24}$$

where $h(\varepsilon_{nn})$ is the Heavyside function. The final form for the friction stress represented with respect to the 12 axes yields,

$$\boldsymbol{\sigma}^f = \begin{bmatrix} 1-h(\varepsilon_{nn}) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-h(\varepsilon_{nn}) \end{bmatrix} [\boldsymbol{T}^{-1}] \int_{-\infty}^{t} \boldsymbol{C}(t-\tau) : \left(\frac{d\boldsymbol{\varepsilon}(\tau)}{d\tau} - \boldsymbol{\varepsilon}^s \right) d\tau. \tag{25}$$

The transformation tensor is a function of the angle formed between the damage components normal to the fiber and matrix directions,

$$\tan\beta = \frac{D_{22}}{D_{11}}. \tag{26}$$

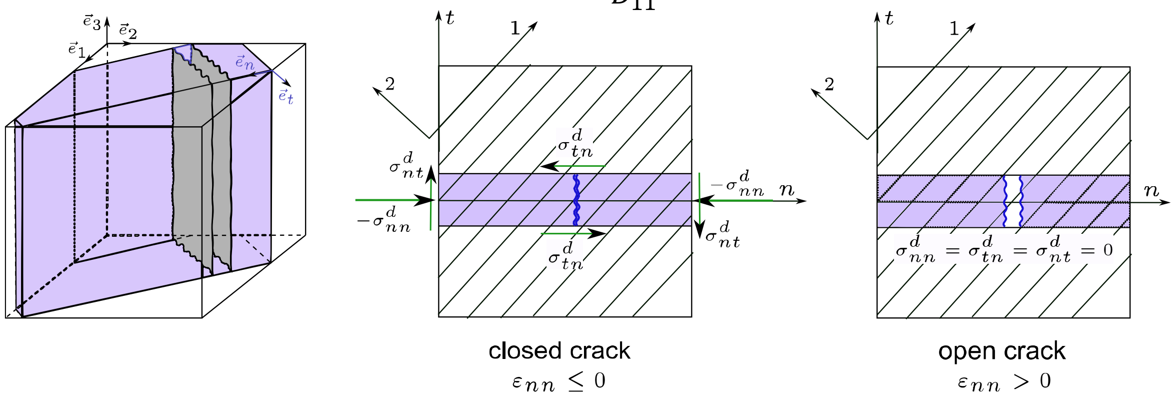

Figure 3. Stress field when crack is closed (left) and open (right).

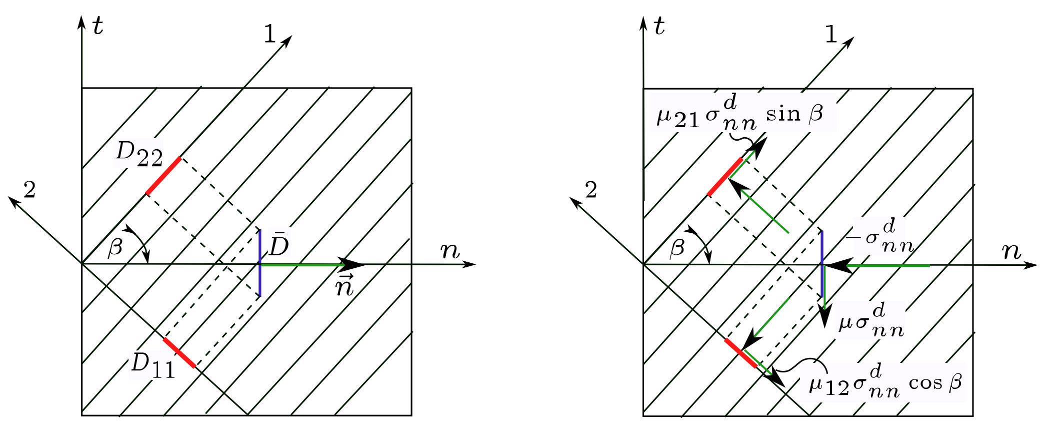

Figure 4. Representation of equivalent damage variable and transformation angle.

3.7. Friction-damage coupling

Some important aspects of the friction phenomena that need to be captured in the model include:

- Friction due to relative sliding of flat crack surfaces only contributes to the shear component of $\boldsymbol{\sigma}$,
- Only the relative sliding of matrix cracks (quantified in D_{12} and D_{21}) acted upon by the shear component of the sliding stress tensor can dissipate energy through friction work.

A first attempt at proposing an expression for the slip stress tensor is given below,

$$\boldsymbol{D}^n(\boldsymbol{\sigma}^f)^n = \begin{bmatrix} D_{11}\sigma_{11}^f & 0 & 0 \\ 0 & D_{22}\sigma_{22}^f & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad 2\boldsymbol{D}^t(\boldsymbol{\sigma}^f)^t(e_1 \otimes e_2)_s = \begin{bmatrix} 0 & D_{12}\sigma_{12}^f & 0 \\ D_{21}\sigma_{12}^f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{27}$$

where the subscript “s” signifies the symmetric part of the tensor. It is assumed that each shear term is activated by the corresponding shear damage term. However, the asymmetry of the tensors of equation (27) requires a symmetrization scheme to be applied,

$$(\boldsymbol{\sigma}^s)^s = \frac{1}{2}(\boldsymbol{\sigma}^s + (\boldsymbol{\sigma}^s)^T) = \begin{bmatrix} D_{11}\sigma_{11}^f & D_{12}^s\sigma_{12}^f & 0 \\ D_{12}^s\sigma_{12}^f & D_{22}\sigma_{22}^f & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{28}$$

It is also important to mention that the sliding stress tensor is linear with respect to the damage terms. This is in accordance with other damage-friction coupled models that consider a scalar-valued damage variable instead [4,16].

3.8. Sliding friction evolution law

The sliding potential is given by,

$$\Phi_s = \mu\sigma_{nn} + \sigma_{nt}. \tag{29}$$

To account for the material anisotropy, friction coefficients along the fiber (μ_{21}) and matrix (μ_{12}) can be introduced [16],

$$\mu\sigma_{nn} = \sqrt{(\sigma_{nn}\mu_{12}\cos\beta)^2 + (\sigma_{nn}\mu_{21}\sin\beta)^2}, \qquad \mu = \sqrt{(\mu_{12}\cos\beta)^2 + (\mu_{21}\sin\beta)^2} \tag{30}$$

The stress terms in equation (29) are converted from the fracture plane to the 12 axes as,

$$\sigma_{nn} = \sigma_{11}^f + \sigma_{22}^f, \qquad \sigma_{nt} = \sin\beta\cos\beta\left(\sigma_{11}^f - \sigma_{22}^f\right) + (\cos^2\beta - \sin^2\beta)\sigma_{12}^f. \tag{31}$$

The same operation can be applied to the strain tensor,

$$\varepsilon_{nn} = \varepsilon_{11}\cos^2\beta + \varepsilon_{22}\sin^2\beta - 2\varepsilon_{12}\sin\beta\cos\beta\,. \tag{32}$$

The sliding evolution law can be derived from the sliding potential as,

$$\dot{\boldsymbol{\varepsilon}}_s = \dot{\lambda}\frac{\partial\Phi_s}{\partial\boldsymbol{\sigma}^s}. \tag{33}$$

3.9. Damage evolution law

This work considers the theory of invariants to define a non-algebraic expression for the damage dissipation potential. For the case of orthotropic scalar-valued functions of a symmetric second-order tensor (i.e. $t = f(\boldsymbol{A})$), the functional basis consists of the following terms [17],

$$\mathrm{tr}\, M_{11}\boldsymbol{A}, \mathrm{tr}\, M_{11}\boldsymbol{A}^{2}, \mathrm{tr}\, \boldsymbol{A}^{3}, \mathrm{tr}\, M_{22}\boldsymbol{A}, \mathrm{tr}\, M_{22}\boldsymbol{A}^{2}, \mathrm{tr}\, M_{33}\boldsymbol{A}, \mathrm{tr}\, M_{33}\boldsymbol{A}^{2}, \tag{34}$$

where $M_{ij} = e_i \otimes e_j, i, j = 1,2,3$. A form of the dissipation potential can be constructed inspired by existing damage evolution models which assume a power-law type [18],

$$\begin{aligned}\Phi_d = & \frac{\alpha_1}{(b+1)(1-D_{11})^a} Y_{11}^{b+1} + \frac{\alpha_2}{(d+1)(1-D_{22})^c} Y_{22}^{d+1} + \frac{\alpha_3}{(f+1)(1-D_{21})^e} Y_{21}^{f+1} \\ & + \frac{\alpha_4}{(d+1)(f+1)(1-D_{21})^e (1-D_{22})^f} Y_{22}^{d+1} Y_{21}^{f+1},\end{aligned} \tag{35}$$

A damage evolution law in the time domain can be obtained as,

$$\dot{\boldsymbol{D}} = \dot{\lambda} \frac{\partial \Phi_d}{\partial \boldsymbol{Y}}, \tag{36}$$

where each term of the left-hand side tensor is given by,

$$\dot{D}_{11} = \frac{\alpha_1}{(1-D_{11})^a} Y_{11}^{b}, \tag{37}$$

$$\dot{D}_{22} = \frac{1}{(1-D_{22})^c} \left(\alpha_2 Y_{22}^{d} + \frac{\alpha_4}{(f+1)(1-D_{21})^e} Y_{22}^{d} Y_{21}^{f+1} \right), \tag{38}$$

$$\dot{D}_{12} = \frac{1}{(1-D_{21})^e} \left(\alpha_3 Y_{21}^{f} + \frac{\alpha_4}{(d+1)(1-D_{22})^c} Y_{22}^{d+1} Y_{21}^{f} \right). \tag{39}$$

3.10. Temperature update

From the first law of thermodynamics, the change in θ at the end of each cycle can be obtained by solving the following differential equation,

$$A_m^s : \dot{V}_m^s - \theta \dot{w}_\theta = -A_k^d : \dot{V}_k^d - \dot{q} - \rho c \dot{\theta}, \tag{40}$$

where $A_m^s : \dot{V}_m^s$ is the power stored in the material due to damage and/or hardening effects, $\dot{w}_\theta$ is the thermomechanical coupling term, $A_k^d : \dot{V}_k^d$ is the dissipated power due to viscous, friction and/or plastic effects, and c is the specific heat.

4. Concluding remarks

A thermodynamically consistent viscoelastic-damage-friction constitutive model has been proposed. Material characterization tests to describe the viscoelastic material behavior and damage evolution laws are still required prior to implementation in a numerical routine. Including a viscoplastic model could reveal beneficial in quantifying dissipation in low frequency conditions.

5. References

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